servatory, Catalogue et Positions observées des Étoiles, vi^h à xii^h, presented by the Observatory; Philosophical Transactions of the Royal Society, vol. i., presented by C. L. Prince; two original negatives of the Moon, presented by the Lick Observatory; Leybourn, W., the Art of Dialling, 1682, Munster, S., Rudimenta Mathematica, 1551, and Sturmy, S., The Mariner's Magazine, 1669, presented by William Schooling.

On the Verification of the Expressions given in Delaunay's Lunar Theory by a Direct Differentiation and Substitution in the Differential Equations. By E. J. Stone, M.A., F.R.S.

In the *Monthly Notices* for November, Professor Cayley has shown by a direct differentiation of the expressions v, u, and $\frac{1}{r}$, given by Delaunay, that they satisfy the differential equations employed to the fourth order of the small fractions $\frac{n'}{n}$ or m, and for terms involving only e; and, if I understand the paper rightly, it is suggested that this method might be applied with success to test the general accuracy of Delaunay's expressions.

The general integration of the three simultaneous differential equations has been carried by Delaunay to the seventh order of the small fractions involved, whilst for certain terms, in which this degree of approximation was found insufficient, special investigations have been undertaken for the determination of the coefficients to much higher orders than the seventh; and the necessary transformations have been effected to give the expressions for v and u complete to the seventh order. But, because the inverse radius vector is ultimately multiplied by a comparatively small linear quantity, the radius of the Earth, and is compared directly with observation under the form of the Lunar Parallax, it has been thought sufficient to give the algebraical expression for $\frac{1}{r}$ only to the fifth order of small quantities; and the coordinates v, u, and $\frac{1}{r}$, as given by Delaunay, are therefore only complete to the fifth order. There would, however, be no very serious difficulty in effecting the necessary transformations and obtaining the expression for $\frac{1}{r}$ to the seventh order of small quantities, and this would render the expressions for v, u, and $\frac{1}{r}$, if accurate, complete to the seventh order. But the labour of extending these expressions generally to the high order of small quantities which has been found necessary for some terms would

be enormous; and this work is hardly likely to be undertaken and carried out. But unless the expressions for the coordinates v, u, and $\frac{1}{r}$ are complete to some definite order, p, it will, I believe, be found impossible to verify the accuracy of the work of integration in series by the inverse process of a direct differentiation of these expressions for v, u, and $\frac{1}{r}$, and the substitution of the results in the differential equations in the manner which has been successfully adopted by Professor Cayley for terms of order e. and to m^4; and if sensible errors exist in Delaunay's expressions they are far more likely to be found in terms of an order higher than the seventh, at which the general integration ends, than amongst those of a lower order.

But, even if the complete integration of the equations employed by Delaunay had been effected, it would be quite impossible to make the functions v, u, and $\frac{1}{r}$ accurately represent the geocentric coordinates of the Moon's centre of gravity by assigning any definite numerical values to the constants a e, γ, τ, g, and h, introduced in the integrations, and of the constant μ, or

$$n=\sqrt{\frac{\mu}{a^3}},$$

because the equations integrated are not the differential equations of motion. And it will be impossible to pass from the integrals v, u, and $\frac{1}{r}$ to the required integrals of the differential equations of motion,

$$v+\delta v,\ u+\delta u, \frac{1}{r}+\ \delta\frac{1}{r}$$

until at least the corrections which the expressions v', u', $\frac{1}{r'}$ adopted for the geocentric coordinates of the Sun are rendered definite, and if the final corrections δv, δu, and $\delta\frac{1}{r}$ are found on the assumption that $\delta v'$ contains no terms of the form

$$\delta(l'+n't)$$

or

$$\delta l'+\delta n'.t+n'\delta t,$$

then the resulting functions,

$$v+\delta v,\ u+\delta u, \frac{1}{r}+\delta\frac{1}{r},$$

will, if complete, be the expressions for the geocentric coordi-

nates of the Moon when t is found from observation, subject to the conditions indicated, that $\delta v'$ contains no terms of the form

$$\delta(l'+n't)$$

or

$$\delta l'+\delta n't+n'.\delta t,$$

but not otherwise.

If such corrections as $\delta l'+\delta n'.t+n'.\delta t$ exist and are sensible, the effects of such terms on the integrals $\delta v.\delta u$. and $\delta.\frac{1}{r}$ must be investigated, and included in the expressions for $v+\delta v.u+\delta u\ \frac{1}{r}+\delta\frac{1}{r}$ before the theoretical results are compared with observation for the determination of the numerical values of the constants involved in their expression. But if $\delta l'+\delta n'.t+n'.\delta t$ is put equal to zero, then the variable t, or the time, must be found subject to the same conditions; and this requires, in order that $\delta t=0$, or the time, t, be found correctly, that $\delta l'=0$ and $\delta n'=0$, or that the epoch from which t is measured, and the unit in which it is expressed, shall be such that the adopted values l' and n' are exact.

If the integrals of the differential equations of motion are correctly found in terms of the variable t, which renders $\delta l'=0$, $\delta n'=0$, or the adopted values of l' and n' exact, and we employ in making the comparisons between theory and observation a value $t+\delta t$ instead of t, the effects of the errors, δt, will be thrown upon the determinations of the constants; and this will be true however the variable $t+\delta t$ may be found from observation, provided it differs from the time, t, on the required scale of time measurements; and if δt is of the form P sin $(pt+q)$, the determinations of the epoch-constants and mean motions will appear liable to periodical changes which cannot be accounted for, quantitatively, by direct theoretical investigations of the motions of the planets and of the Moon, based on the usual assumption that $\delta(l'+n't)=0$. There will appear in such cases a necessity for empirical long inequalities to secure an agreement between the tabular results and those of observation. But such empirical corrections should, I believe, be most carefully avoided, and attention directed to the theoretical inequalities in solar theory to avoid errors, δt, and to the completion of the theories of the planets and Moon on the present assumption that $\delta(l'+n't)=0$. It may be mentioned that the right ascensions of the stars are found subject to the same condition, and that from them the right ascensions of the meridians are interpolated by the aid of clocks, and rendered definite by direct references to the Sun's positions at the meridian passages.

On the Determination of a certain Class of Inequalities in the Moon's Motion. By Ernest W. Brown, B.A.

1. The two principal methods of treating the Lunar Theory—viz. (1) the *general* method, in which we obtain a general approximate solution, applying to any single Moon whose present orbit lies within given limits, in terms of certain constants, the coordinates of position and the time; and (2) the *special* method, in which we use the numerical values of those constants at the outset for any given Moon, and obtain a solution involving the coordinates of position and the time only—present each of them certain advantages. In the first case, we have the theory, worked out to a certain degree of accuracy, immediately applicable to any single Moon in our solar system, and therefore arranged in such a way that any small change which improved data may involve in the values of the constants can be made easily without requiring us to go over the whole of the work again. On the other hand, the number of terms which have been found necessary to secure a degree of accuracy commensurate with that of observation is very large, and it becomes a task of great labour to obtain them with any degree of certainty. In the second case, when we use numerical values from the start, the portions arising from the expansion of the functions in ascending powers of the constants involved, which portions in the first case would have to be neglected, are no longer left out of account. A great increase of accuracy naturally results. But this method is unsatisfactory in some respects. Having started with certain values of the constants for any particular Moon, we are bound to keep to those values; and also if any numerical mistake be made, it is not very easily traceable. The calculation is applicable only to the particular Moon selected. Any method, therefore, of finding the path of the Moon which will give at the same time numerical and algebraical results with equal accuracy would be of value.

Mr. G. W. Hill, in the *American Journal of Mathematics*, vol. i.,* has shown how to separate out those inequalities dependent *only* on the ratio of the mean motions of the Sun and Moon, and to obtain their coefficients both numerically and as an algebraic function in ascending powers of this ratio. He shows also that his method is not only susceptible of a very high degree of accuracy with comparatively little trouble, but that any higher degree of accuracy can be obtained without the necessity of going over a large part of the ground again. His method for this class of inequalities is as follows:—

2. Take as origin the centre of the Earth considered as a sphere, and refer the motion to rectangular axes revolving round the origin with the mean angular velocity n' of the Sun round the Earth. If we neglect all the differences from purely elliptic

* *Researches in the Lunar Theory*, pp. 5, 129, 245